

POINCARÉ DUALITY COMPLEXES WITH HIGHLY CONNECTED UNIVERSAL COVERS

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ABSTRACT. Baues and Bleile showed that, up to oriented homotopy equivalence, a Poincaré duality complex of dimension $n \geq 3$ with $(n-2)$ -connected universal cover, is classified by its fundamental group, orientation class and the image of its fundamental class in the homology of the fundamental group. We generalise Turaev's results for the case $n = 3$, by providing necessary and sufficient conditions for a triple (G, ω, μ) , comprising a group, G , $\omega \in H^1(G; \mathbb{Z}/2\mathbb{Z})$ and $\mu \in H_3(G; \mathbb{Z}^\omega)$ to be realised by a Poincaré duality complex of dimension n with $(n-2)$ -connected universal cover, and by showing that such a complex is a connected sum of two such complexes if and only if its fundamental group is a free product of groups.

1. INTRODUCTION

Important properties of manifolds, especially global ones such as *Poincaré duality*, depend only on their homotopy properties. This realisation led to the notion of *Poincaré duality complexes* as the homotopy theoretic generalisation of manifolds, opening the possibility of classifying manifolds using a classification of Poincaré duality complexes.

Since not every Poincaré duality complex arises from a manifold, such complexes are genuinely more general. In this case, the greater generality is not an additional complication, but a convenience, for Poincaré duality complexes are accessible to algebraic analysis, thereby allowing more computation. This has been successfully exploited by a number of authors to achieve partial classification using algebraic invariants.

In particular, Hendricks classified Poincaré duality complexes of dimension 3 (*PD³-complexes*) up to oriented homotopy equivalence using the “fundamental triple”, comprising the fundamental group, the orientation character and the image of the fundamental class in the cohomology of the fundamental group.

Turaev ([5]) gave an alternative proof of Hendricks' result and provided necessary and sufficient conditions for a triple (G, ω, μ) , consisting of a group, G , a cohomology class, $\omega \in H^1(G; \mathbb{Z}/2\mathbb{Z})$, and a homology class, $\mu \in H_3(G; \mathbb{Z}^\omega)$, where \mathbb{Z}^ω denotes the integers, \mathbb{Z} , regarded as a right $\mathbb{Z}[G]$ -module with respect to the *twisted* structure induced by ω , to be fundamental triple of a PD^3 -complex with simply connected universal covering space. Central to this was that applying a specific function, which we call *the Turaev map*, to μ , yield isomorphism in the stable category of $\mathbb{Z}[G]$, that is to say, a homotopy equivalence of $\mathbb{Z}[G]$ -modules.

Turaev conjectured that his result holds for all PD^n -complexes with $(n-2)$ -connected universal cover.

Baues and Bleile classified Poincaré duality complexes of dimension 4 in [2]. Their analysis showed that a Poincaré duality complex, X , of dimension $n \geq 3$, is classified up to oriented homotopy equivalence by the triple comprising its $(n-2)$ -type $P_{n-2}(X)$, its orientation character $\omega_X \in H^1(\pi_1(X); \mathbb{Z}/2\mathbb{Z})$ and its fundamental class, $[X] \in H_n(X; \mathbb{Z}^{\omega_X})$.

They called this the *fundamental triple of X* , as it is a generalisation of Hendricks' “fundamental triple”, for the $(n-2)$ type of X determines its fundamental group. Moreover, when the universal cover of the complex is $(n-2)$ -connected — automatically the case when $n = 3$ — the $(n-2)$ section is an Eilenberg-Mac Lane space of type $(\pi_1(X), 1)$, so that the $(n-2)$ type and the fundamental group determine each other completely, reducing their fundamental triple to that of Hendricks.

It follows that two PD^n -complexes with $(n-2)$ -connected universal cover are homotopically equivalent if and only if their fundamental triples are isomorphic, verifying the first part of Turaev's conjecture.

Our main results complete a proof of Turaev's conjecture, which we formulate in two theorems.

Recall that the group G is of type FP_n if and only if the trivial $\mathbb{Z}[G]$ -module \mathbb{Z} has a projective resolution, \mathbf{P} , with P_j finitely generated for $j \leq n$.

Theorem A. *Let G be a finitely presentable group, ω a cohomology class in $H^1(G; \mathbb{Z}/2\mathbb{Z})$, and μ a homology class in $H_n(G; \mathbb{Z}^\omega)$.*

If G is of type FP_n with $H^i(G; {}^\omega\mathbb{Z}[G]) = 0$ for $1 < i < n-1$, and $n \geq 3$, then (G, ω, μ) can be realised as the fundamental triple of a PD^n -complex with $(n-2)$ -connected universal cover if and only if the Turaev map applied to μ is an isomorphism in the stable category of $\mathbb{Z}[G]$.

Theorem B. *A PD^n -complex with $(n-2)$ -connected universal cover, decomposes as connected sum if and only if its fundamental group decomposes as a free product of groups.*

Section 2 summarises background material and fixes notation.

Section 3 contains the formulation and proof of the necessity of the condition for the realisation of a fundamental triple by a PD^n -complex with $(n-2)$ -connected universal cover.

Section 4 completes the proof of Theorem A, with the sufficiency of the condition in Section 3.

Section 5 contains the proof of Theorem B, showing how the fundamental triple detects connected sums.

2. BACKGROUND AND NOTATION

This section summarises background material and fixes notation for the rest of the paper. Details and further references can be found in [3].

2.1. Let Λ be the integral group ring, $\mathbb{Z}[G]$, of the finitely presented group, G .

We write I for the *augmentation ideal*, the kernel of the augmentation map

$$\text{aug}: \Lambda \longrightarrow \mathbb{Z}, \quad \sum_{g \in G} n_g g \longmapsto \sum_{g \in G} n_g$$

where \mathbb{Z} is a Λ -bi-module with trivial Λ action.

Each cohomology class $\omega \in H^1(G; \mathbb{Z}/2\mathbb{Z})$ may be viewed as a group homomorphism

$$\omega: G \longrightarrow \mathbb{Z}/2\mathbb{Z} = \{0, 1\}$$

and yields an anti-isomorphism

$$\bar{}: \Lambda \longrightarrow \Lambda, \quad \lambda = \sum_{g \in G} n_g g \longmapsto \bar{\lambda} = \sum_{g \in G} (-1)^{\omega(g)} n_g g^{-1}$$

Consequently, a right Λ -module, A , yields the left Λ -module, ${}^\omega A$, with action given by

$$\lambda \cdot a := a \cdot \bar{\lambda}$$

for $\lambda \in \Lambda, a \in A$.

Analogously, a left Λ -module B yields the right Λ -module, B^ω .

Given left Λ -modules A_j, B_i for $1 \leq i \leq k$ and $1 \leq j \leq \ell$, we sometimes write the Λ -morphism

$$\psi: \bigoplus_{j=1}^{\ell} A_j \longrightarrow \bigoplus_{i=1}^k B_i$$

in matrix form as

$$\bigoplus_{j=1}^{\ell} A_j \xrightarrow{[\psi_{ij}]_{k \times \ell}} \bigoplus_{i=1}^k B_i$$

for $\psi_{ij} = pr_i \circ \psi \circ in_j: A_j \longrightarrow B_i$, where in_j is the j^{th} natural inclusion and pr_i the i^{th} natural projection of the direct sum. The composition of such morphisms is given by matrix multiplication.

If B is a left Λ -module and M a Λ -bi-module, then $\text{Hom}_\Lambda(B, M)$ is a right Λ -module with action given by

$$\varphi \cdot \lambda: B \longrightarrow M, \quad b \longmapsto \varphi(b) \cdot \lambda$$

The dual of the left Λ -module B is the left Λ -module $B^* = {}^\omega \text{Hom}_\Lambda(B, \Lambda)$.

This defines an endofunctor on the category of left Λ -modules.

Evaluation defines a natural transformation, ω_ε , from the identity functor to the double dual functor, where for the left Λ -module B ,

$$\omega_\varepsilon_B: B \longrightarrow B^{**} = {}^\omega \text{Hom}_\Lambda({}^\omega \text{Hom}_\Lambda(B, \Lambda), \Lambda), \quad b \longmapsto \overline{\text{ev}_b}$$

with $\overline{\text{ev}_b}$ defined by

$$\overline{\text{ev}_b}: {}^\omega \text{Hom}_\Lambda(B, \Lambda) \longrightarrow \Lambda, \quad \psi \longmapsto \overline{\psi(b)}$$

The left Λ -module, A , defines the natural transformation, η , from the functor $A^\omega \otimes_\Lambda _$ to the functor $\text{Hom}_\Lambda({}^\omega \text{Hom}_\Lambda(_, \Lambda), A)$ where, for the left Λ -module B ,

$$\eta_B: A^\omega \otimes_\Lambda B \longrightarrow \text{Hom}_\Lambda(B^*, A) = \text{Hom}_\Lambda({}^\omega \text{Hom}_\Lambda(B, \Lambda), A)$$

is given by

$$\eta_B(a \otimes b): \psi \longmapsto \overline{\psi(b)} \cdot a$$

for $a \otimes b \in A^\omega \otimes_\Lambda B$.

The Λ -morphisms $f, g: A_1 \longrightarrow A_2$ are *homotopic* if and only if the homomorphism

$$f - g: A_1 \longrightarrow A_2, \quad x \longmapsto f(x) - g(x)$$

factors through a projective Λ -module P .

Associated with Λ is its *stable category*, whose objects are all Λ -modules and whose morphisms are all homotopy classes of Λ -morphisms. Thus, an isomorphism in the stable category of Λ is a homotopy equivalence of Λ -modules.

2.2. We work in the category of connected well pointed CW -complexes and pointed maps. We write $X^{[k]}$ for the k -skeleton of X , suppressing the base point from our notation.

The inclusion of the k -skeleton into the $(k+1)$ -skeleton induces homomorphisms

$$\iota_{k,r}: \pi_r(X^{[k]}) \longrightarrow \pi_r(X^{[k+1]})$$

and we write $\Gamma_{k+1}(X)$ for $\text{im}(\iota_{k,k+1})$.

From now, we work with the fundamental group of X , and its integral group ring

$$\Lambda = \mathbb{Z}[\pi_1(X)]$$

We take X to be a reduced CW -complex, so that $X^{[0]} = \{*\}$, and write

$$u: \tilde{X} \longrightarrow X$$

for the universal cover of X , fixing a base point for \tilde{X} in $u^{-1}(*)$.

We write $\mathbf{C}(\tilde{X})$ for the cellular chain complex of \tilde{X} viewed as a complex of left Λ -modules.

Since X is reduced, $C_0(\tilde{X}) = \Lambda$.

The homology and cohomology of X we work with are the abelian groups

$$\begin{aligned} \hat{H}_q(X; A) &:= H_q(A \otimes_\Lambda \mathbf{C}(\tilde{X})) \\ \hat{H}^q(X; B) &:= H^q(\text{Hom}_\Lambda(\mathbf{C}(\tilde{X}), B)) \end{aligned}$$

where A is a right Λ -module and B is a left Λ -module.

We write $H_{-q}(X; B)$ for $H^q(X; B)$, when treating cohomology as “homology in negative degree” simplifies the exposition.

2.3. An n -dimensional Poincaré duality complex (PD n -complex) comprises a reduced connected CW-complex, X , whose fundamental group, $\pi_1(X)$, is finitely presented, together with an *orientation character*, $\omega_X \in H^1(\pi_1(X); \mathbb{Z}/2\mathbb{Z})$, viewed as a group homomorphism $\pi_1(X) \rightarrow \mathbb{Z}/2\mathbb{Z}$, and a *fundamental class* $[X] \in H_n(X; \mathbb{Z}^{\omega_X})$, such that for every $r \in \mathbb{Z}$ and left $\mathbb{Z}[\pi_1(X)]$ -module M , the *cap product with* $[X]$,

$$- \cap [X]: H^r(X; M) \rightarrow H_{n-r}(X; M^{\omega_X}), \quad \alpha \mapsto \alpha \cap [X]$$

is an isomorphism of abelian groups.

We denote this by $(X, \omega_X, [X])$.

Wall ([6], [8]) showed that for $n > 3$, every PD n complex is *standard*, meaning that it is homotopically equivalent to an n -dimensional CW-complex with precisely one n -cell, whereas a PD 3 complex, X , is either standard, or *weakly standard*, meaning that it is homotopically equivalent to one of the form $X' \cup e^3$, where e^3 is a 3-cell and X' is a 3-dimensional CW-complex with $H^n(X'; B) = 0$ for all coefficient modules B .

2.4. **Homotopy Systems.** Baues introduced *homotopy systems* to investigate when chain complexes and chain maps of free Λ -modules are realised by CW-complexes ([1]).

Definition 2.1. Let n be an integer greater than 1. A *homotopy system of order* $(n+1)$ comprises

- (a) a reduced n -dimensional CW-complex, X ,
- (b) a chain complex \mathbf{C} of free $\Lambda := \mathbb{Z}[\pi_1(X)]$ -modules which coincides with $\mathbf{C}(\tilde{X})$ in degree q whenever $q \leq n$,
- (c) a homomorphism, $f_{n+1}: C_{n+1} \rightarrow \pi_n(X)$ with $f_{n+1} \circ d_{n+2} = 0$, which renders commutative the diagram

$$\begin{array}{ccc} C_{n+1} & \xrightarrow{f_{n+1}} & \pi_n(X) \\ d_{n+1} \downarrow & & \downarrow j \\ C_n & \xleftarrow{h_n} & \pi_n(X, X^{[n-1]}) \end{array}$$

where j is induced by the inclusion $(X, *) \rightarrow (X, X^{[n-1]})$, and

$$h_n: \pi_n(X, X^{[n-1]}) \xrightarrow[\cong]{u_*^{-1}} \pi_n(\tilde{X}, \tilde{X}^{[n-1]}) \xrightarrow[\cong]{h} H_n(\tilde{X}, \tilde{X}^{[n-1]})$$

is the Hurewicz isomorphism, h , composed with u_*^{-1} , the inverse of the isomorphism induced by the universal covering map.

3. FORMULATION AND NECESSITY OF THE REALISATION CONDITION

3.1. As the $(n-2)$ -type, $P_{n-2}X$, of the PD n -complex, X , with $(n-2)$ -connected universal cover, is an Eilenberg-Mac Lane space, $K(\pi_1(X), 1)$, we may identify the fundamental triple of X with $(\pi_1(X), \omega, \mu)$, where ω and μ are, respectively, the images of ω_X and $[X]$ in the group homology of $\pi_1(X)$.

Lemma 3.1. Let $(X, \omega_X, [X])$ be a PD n -complex with $(n-2)$ -connected universal cover. Then, for all $1 \leq i < n-1$,

$$H^i(\pi_1(X); \omega\Lambda) = 0$$

Proof. Construct an Eilenberg-Mac Lane space $K = K(\pi_1(X), 1)$ from X by attaching cells of dimension n and higher.

As the universal cover \tilde{X} of X is $(n-2)$ -connected, the cellular chain complexes of the universal covers \tilde{X} and \tilde{K} coincide in degrees below n :

$$C_i(\tilde{X}) = C_i(\tilde{K}) \quad \text{for } 0 \leq i < n$$

Hence, for $1 < i < n - 1$,

$$\begin{aligned}
H^i(\pi_1(X); \omega\Lambda) &= H^i(\tilde{K}; \omega\Lambda) \\
&= H^i(\tilde{X}; \omega\Lambda) \\
&\cong H_{n-i}(\tilde{X}; \Lambda) && \text{by Poincaré duality} \\
&= 0
\end{aligned}$$

□

3.2. To formulate the realisation condition, we introduce set of functors, $\{G_q \mid q \in \mathbb{Z}\}$, from the category of chain complexes of projective left Λ -modules and homotopy classes of chain maps to the stable category of Λ .

Given chain complexes of projective left Λ -modules, \mathbf{C}, \mathbf{D} , and a chain map $\mathbf{f}: \mathbf{C} \rightarrow \mathbf{D}$, we have the commutative diagram

$$\begin{array}{ccccccc}
\cdots & \longrightarrow & C_q & \xrightarrow{d_q^{\mathbf{C}}} & C_{q-1} & \longrightarrow & \cdots \\
& & \downarrow f_q & & \downarrow f_{q-1} & & \\
\cdots & \longrightarrow & D_q & \xrightarrow{d_q^{\mathbf{D}}} & D_{q-1} & \longrightarrow & \cdots
\end{array}$$

Put

$$G_q(\mathbf{C}) := \text{coker}(d_q^{\mathbf{C}}) = C_{q-1} / \text{im}(d_q^{\mathbf{C}})$$

and let $G_q(\mathbf{f})$ be the induced map of cokernels

$$\begin{array}{ccccc}
\text{im}(d_q^{\mathbf{C}}) & \xrightarrow{\quad} & C_{q-1} & \twoheadrightarrow & G_q(\mathbf{C}) \\
\downarrow & & \downarrow f_{q-1} & & \downarrow \exists! G_q(\mathbf{f}) \\
\text{im}(d_q^{\mathbf{D}}) & \xrightarrow{\quad} & D_{q-1} & \twoheadrightarrow & G_q(\mathbf{D})
\end{array}$$

Direct verification shows that each G_q is a functor from the category of chain complexes of left Λ -modules to the category of left Λ -modules.

By Lemma 4.2 in [3], chain homotopic maps $\mathbf{f} \simeq \mathbf{g}: \mathbf{C} \rightarrow \mathbf{D}$ induce homotopic maps $G_n(\mathbf{f}) \simeq G_n(\mathbf{g})$, that is, $G_q(\mathbf{f}) - G_q(\mathbf{g})$ factors through a projective Λ -module.

Hence, for each $q \in \mathbb{Z}$, G_q is a functor from the category of chain complexes of projective left Λ -modules to the stable category of Λ .

3.3. Let X be PD^n complex with $n \geq 3$. Put $\Lambda = \mathbb{Z}[\pi_1(X)]$ and take $\omega \in H^1(\pi_1(X); \mathbb{Z}/2\mathbb{Z})$.

By Remark 2.3 and Lemma 3.6 in [2], we may assume that $X = X' \cup e^n$ is standard (or weakly standard if $n = 3$) with

$$C_n(\tilde{X}) = C_n(\tilde{X}') \oplus \Lambda[e]$$

where $[e]$ corresponds to e^n , $1 \otimes [e] \in \mathbb{Z}^\omega \otimes_\Lambda C_n(\tilde{X})$ is a cycle representing $[X]$, the fundamental class of X , and $[e]$ is a generator of $C_n(\tilde{X})$.

Write F^q for $G_q({}^\omega \text{Hom}_\Lambda(_, \Lambda))$.

Poncaré duality, together with Lemma 4.3 in [3], provides the homotopy equivalence of Λ -modules

$$G_{-n+1}(\underline{\quad} \frown (1 \otimes [e])): F^{n-1}(\mathbf{C}(\tilde{X})) \rightarrow G_1(\mathbf{C}(\tilde{X}))$$

Since $\hat{H}_1(C(\tilde{X})) = 0$, $G_1(\mathbf{C}(\tilde{X})) = I$. Thus,

$$\theta: G_1(\mathbf{C}(\tilde{X})) \rightarrow I, \quad [c] \mapsto d_1(c)$$

is an isomorphism of Λ -modules.

Construct the $(n-2)$ -type of X , $P = P_{n-2}(X)$ by attaching to X cells of dimension n and higher.

Then the Postnikov section $p: X \rightarrow P$ is the identity on the $(n-1)$ -skeleta and for $0 \leq i < n$,

$$C_i(\tilde{X}) = C_i(\tilde{P})$$

Composing G_{-n+1} with θ , we obtain the homotopy equivalence of left Λ -modules

$$\theta \circ G_{-n+1}(_ \frown (1 \otimes [e])): F^{n-1}(\mathbf{C}(P)) \rightarrow I$$

3.4. Given a chain complex of free left Λ -modules, \mathbf{C} , we define a homomorphism

$$\nu_{\mathbf{C},r}: H_{r+1}(\mathbb{Z}^\omega \otimes_\Lambda \mathbf{C}) \rightarrow [F^r(\mathbf{C}), I]$$

with the property that $\nu_{\mathbf{C}(\tilde{P}),n-1}(u_*([X]))$ is the homotopy class of the homotopy equivalence

$$\theta \circ G_{-n+1}(_ \frown (1 \otimes [e])): F^{n-1}(\mathbf{C}(\tilde{X})) \rightarrow I$$

The short exact sequence of chain complexes

$$0 \rightarrow \bar{I}\mathbf{C} \rightarrow \mathbf{C} \rightarrow \mathbb{Z}^\omega \otimes_\Lambda \mathbf{C} \rightarrow 0$$

yields the connecting homomorphism

$$\begin{aligned} \delta: H_{r+1}(\mathbb{Z}^\omega \otimes_\Lambda \mathbf{C}) &\rightarrow H_r(\bar{I}\mathbf{C}) \\ 1 \otimes c &\mapsto d_{r+1}(c) \end{aligned}$$

Then $\nu_{\mathbf{C},r}$ is the composition of δ with

$$\hat{\nu}_{\mathbf{C},r}: H_r(\bar{I}\mathbf{C}) \rightarrow [F^r(\mathbf{C}), I]$$

where, for $[\lambda.c] \in H_r(\bar{I}\mathbf{C})$, $\tilde{\nu}_{\mathbf{C},r}([\lambda.c])$ is the class of

$$[\varphi] \mapsto \overline{\varphi(\lambda.c)} = \overline{\varphi(c)}.\bar{\lambda}$$

Lemma 3.2. *The map $\tilde{\nu}_{\mathbf{C},r}$ is a well defined homomorphism of groups.*

Proof. Let $\lambda.c \in \bar{I}\mathbf{C}$ be a cycle.

Since $\text{aug}(\bar{\lambda}) = 0$,

$$\text{aug}(\overline{\varphi(c)}.\bar{\lambda}) = \overline{\varphi(c)}.\text{aug}(\bar{\lambda}) = 0$$

showing that $\overline{\varphi(c)}.\bar{\lambda} \in I$.

Further, for $\varphi' = \varphi + d_r^* \psi = \varphi + \psi d_r$, we obtain

$$\begin{aligned} \overline{\varphi'(\lambda.c)} &= \overline{\varphi(\lambda.c)} + \overline{\psi(d_r(\lambda.c))} \\ &= \overline{\varphi(\lambda.c)} + 0 \\ &= \overline{\varphi(\lambda.c)} \end{aligned}$$

showing that $\overline{\varphi(c)}.\bar{\lambda}$ depends only on $[\varphi]$, not φ itself.

Finally, for $\lambda \in \bar{I}$, $c \in C_{r+1}$, $\tilde{\nu}_{\mathbf{C},r}([d_{r+1}(\lambda.c)])$ is represented by the composition of the Λ -morphisms

$${}^\omega\text{Hom}_\Lambda(C_r, \Lambda) \rightarrow \Lambda, \quad \varphi \mapsto \overline{\varphi(d_{r+1}c)}$$

followed by

$$\varrho_{\bar{\lambda}}: \Lambda \rightarrow I, \quad \mu \mapsto \mu.\bar{\lambda}$$

As Λ is free, this composition is null-homotopic, showing that $\tilde{\nu}_{\mathbf{C},r}([\lambda.c])$ is independent of the representative of the class $[\lambda.c]$.

Thus $\hat{\nu}$ is well defined. □

The *Turaev map* is the morphism $\nu_{\mathbf{C}(\tilde{P}_{n-2}),n-1}$.

Lemma 3.3. $\nu_{\mathbf{C}(\tilde{P}),n-1}(p_*([X])) = [\theta \circ G_{-n+1}(_ \frown (1 \otimes [e]))]$

Proof. Take a diagonal $\Delta: \mathbf{C}(\tilde{X}) \longrightarrow \mathbf{C}(\tilde{X}) \otimes_{\mathbb{Z}} \mathbf{C}(\tilde{X})$ and a chain homotopy $\alpha: \mathbf{C}(\tilde{X}) \longrightarrow \mathbf{C}(\tilde{X})$ such that

$$id - (id \otimes \text{aug})\Delta = d\alpha + \alpha d$$

where we identify $\mathbf{C} \otimes_{\mathbb{Z}} \mathbb{Z}$ with \mathbf{C} .

Suppose that

$$\Delta e = e \otimes \lambda + \sum_{\ell} \sum_{0 \leq i < n} x_{\ell, i} \otimes y_{\ell, n-i}$$

Then

$$\begin{aligned} e &= (id \otimes \text{aug}) \left(e \otimes \lambda + \sum_{\ell} \sum_{0 \leq i < n} x_{\ell, i} \otimes y_{\ell, n-i} \right) + d\alpha e + \alpha d e \\ &= (id \otimes \text{aug}) \left(e \otimes \lambda + \sum_{\ell} \sum_{0 \leq i < n} x_{\ell, i} \otimes y_{\ell, n-i} \right) + 0 + \alpha d e && \text{as } C_{n+1}(\tilde{X}) = 0 \\ &= e \otimes \text{aug}(\lambda) + \alpha d e \\ &= \text{aug}(\lambda)e + \alpha d e \end{aligned}$$

where we have identified $C_n(\tilde{X}) \otimes C_0(\tilde{X}) = C_n(\tilde{X}) \otimes \Lambda$ with $C_n(\tilde{X})$, using $c \otimes \lambda \longmapsto \lambda.e$.

Since $[1 \otimes e]$ generates $\hat{H}_n(X; \mathbb{Z}^{\omega}) \cong \mathbb{Z}$,

$$\begin{aligned} 1 \otimes e &= 1 \otimes \text{aug}(\lambda.e) + 1 \otimes \alpha d e \\ &= \text{aug}(\lambda) \otimes e + (1 \otimes \alpha)(1 \otimes d)(1 \otimes e) \\ &= \text{aug}(\lambda) \otimes e + (1 \otimes \alpha)(0) \end{aligned}$$

whence $\text{aug}(\lambda) = 1$, or $\text{aug}(\lambda - 1) = 0$.

Hence, $\lambda - 1 \in I = \text{im}(d_2)$, or, $\lambda = 1 + d_2(c_2)$ for some $c_2 \in C_2(\tilde{X})$.

Thus, given $\varphi \in \text{Hom}_{\Lambda}(C_n(\tilde{X}), \Lambda)$,

$$\begin{aligned} \varphi \circ (1 \otimes e) &= \varphi(e) \otimes \lambda \\ &= \varphi(e) \otimes (1 + d_2(c_2)) \\ &= \overline{\varphi(e)}(1 + d_2(c_2)) \end{aligned}$$

Since $\mathbf{G}(_ \circ (1 \otimes [e]))$ is a chain map,

$$\begin{aligned} \left(\theta \circ G_{-n+1}(_ \circ (1 \otimes [e])) \right) ([\varphi]) &= d_1 \left(G_{-n+1}(_ \circ (1 \otimes [e])) ([\varphi]) \right) \\ &= G_{-n}(_ \circ (1 \otimes [e])) \left([d_n^*(\varphi)] \right) \\ &= d_n^*(\varphi) \circ (1 \otimes [e]) \\ &= \overline{d_n^*(\varphi)(e)}(1 + dc_2 d_2) \\ &= \overline{\varphi(d_n(e))}(1 + c_2 d_2) \end{aligned}$$

On the other hand,

$$\begin{aligned} \nu_{\mathbf{C}(\tilde{P}), n-1}(p_*([X]))([\varphi]) &= \nu_{\mathbf{C}(\tilde{P}), n-1}(p_*(1 \otimes [e]))([\varphi]) \\ &= \hat{\nu}_{\mathbf{C}(\tilde{P}), n-1}(\delta(1 \otimes [e]))([\varphi]) \\ &= \hat{\nu}_{\mathbf{C}(\tilde{P}), n-1}([d_n(e)])([\varphi]) \end{aligned}$$

Hence, by definition, $\hat{\nu}_{\mathbf{C}(\tilde{P}), n-1}(p_*([X]))$ is represented by the Λ -morphism

$$F^{n-1}(\mathbf{C}(\tilde{P})) \longrightarrow I, \quad [\varphi] \longmapsto \overline{\varphi(d_n(e))}$$

To conclude the proof, note that

$$F^{n-1}\mathbf{C}(\tilde{P}) \longrightarrow I, \quad [\varphi] \longmapsto \overline{\varphi(d_n(e))}.d_2(c_2)$$

is null-homotopic. □

3.5. As $\theta \circ G_{-n+1}(_ \smallfrown (1 \otimes [e]))$ is a homotopy equivalence of Λ -modules, Lemma 3.3 provides a necessary condition for realisation.

Theorem 3.4. *If a fundamental triple (P_{n-2}, ω, μ) , consisting of an $(n-2)$ -type, $P_{n-2}, \omega \in H^1(P_{n-2}; \mathbb{Z}/2\mathbb{Z})$ and $\mu \in H_n(P_{n-2}; \mathbb{Z}^\omega)$, is the fundamental triple of a PD^n -complex X , then $\nu_{\mathbf{C}(\tilde{P}_{n-2}), n-1}(\mu)$, the Turaev map applied to μ , is a homotopy equivalence of modules over $\Lambda = \mathbb{Z}[\pi_1(P_{n-2})]$.*

Proof. Let P_{n-2} be an $(n-2)$ -type. Take $\omega \in H^1(P_{n-2}; \mathbb{Z}/2\mathbb{Z})$ and $\mu \in H_n(P_{n-2}; \mathbb{Z}^\omega)$. Suppose that (P_{n-2}, ω, μ) is the fundamental triple of the PD^n -complex, X .

If P'_{n-2} is an $(n-2)$ -type obtained by attaching to X cells of dimension n and higher, then there is a homotopy equivalence $f: P_{n-2} \rightarrow P'_{n-2}$ with $f_*(\mu) = i_*([X])$, where $i: X \rightarrow P'_{n-2}$ is the inclusion.

By Lemma 3.3,

$$\nu_{\mathbf{C}(\tilde{P}'_{n-2}), n-1}(i_*[X]) = [\theta \circ G_{-n+1}(_ \smallfrown (1 \otimes [e]))]$$

and is hence a homotopy equivalence of Λ -modules.

The diagram

$$\begin{array}{ccc} F^{n-1}(\mathbf{C}(\tilde{P}_{n-2})) & \xrightarrow{\nu_{\mathbf{C}(\tilde{P}_{n-2}), n-1}(\mu)} & I \\ F^{n-1}(f_*) \downarrow & & \parallel \\ F^{n-1}(\mathbf{C}(\tilde{P}'_{n-2})) & \xrightarrow{\nu_{\mathbf{C}(\tilde{P}'_{n-2}), n-1}(f_*(\mu)) = \nu_{\mathbf{C}(\tilde{P}'_{n-2}), n-1}(i_*[X])} & I \end{array}$$

commutes, so that $\nu_{\mathbf{C}(\tilde{P}_{n-2}), n-1}(\mu)$ is a homotopy equivalence of modules. \square

Necessary conditions for realisation are a corollary to Lemma 3.1 and Theorem 3.4

Corollary 3.5 (Conditions for Realisability). *Let (G, ω, μ) be a triple with G a group, $\omega \in H^1(G; \mathbb{Z}/2\mathbb{Z})$ and $\mu \in H_n(G; \mathbb{Z}^\omega)$.*

If (G, ω, μ) is the fundamental triple of a PD^n -complex, X , with $(n-2)$ -connected universal cover, then $\nu_{\mathbf{C}(\tilde{K}), n-1}(\mu)$ is a homotopy equivalence of $\Lambda = \mathbb{Z}[\pi_1(X)]$ modules, and for $1 < i < n-1$

$$H^i(G; {}^\omega\Lambda) = 0$$

4. SUFFICIENCY OF THE REALISATION CONDITION

Let G be a finitely presentable group of type FP_n , with $H^i(G; {}^\omega\mathbb{Z}[G]) = 0$ for $1 < i < n-1$, and $n \geq 3$.

Let $K' = K(G, 1)$ be an Eilenberg-Mac Lane space and identify the (co-)homologies of G and K' .

Given $\omega \in H^1(G; \mathbb{Z}/2\mathbb{Z})$ and $\mu \in H_n(G; \mathbb{Z}^\omega)$, with $\nu_{\mathbf{C}(\tilde{K}'), n-1}(\mu)$ a class of homotopy equivalences of $\mathbb{Z}[G]$ -modules, we construct a PD^n -complex, X , with $(n-2)$ -connected universal cover and fundamental triple (G, ω, μ) .

Let $\tilde{K}' \rightarrow K'$ be the universal covering space of K' .

By our assumptions about G , we can choose K' with finitely many cells in each dimension.

Let $h: F^{n-1}(\mathbf{C}(\tilde{K}')) \rightarrow I$ be a representative of $\nu_{\mathbf{C}(\tilde{K}'), n-1}(\mu)$. Then h is a homotopy equivalence of Λ -modules. By Theorem 4.1 and Observation 1 in [3], h factors as

$$F^{n-1}(\mathbf{C}(\tilde{K}')) \twoheadrightarrow F^{n-1}(\mathbf{C}(\tilde{K}')) \oplus \Lambda^m \xrightarrow{j} I \oplus P \twoheadrightarrow I$$

for some $m \in \mathbb{N}$ and P a projective Λ -module.

Let $B = (e^0 \cup e^{n-1}) \cup e^n$ be the n -dimensional ball and replace K' by the Eilenberg-Mac Lane space

$$K = K' \cup \left(\bigcup_{i=1}^m B \right).$$

Then $F^{n-1}(\mathbf{C}(\tilde{K})) = F^{n-1}(\mathbf{C}(\tilde{K}')) \oplus \Lambda^m$ and the factorisation of h becomes

$$h: F^{n-1}(\mathbf{C}(\tilde{K})) \xrightarrow{j} I \oplus P \xrightarrow{pr_I} I$$

Consider the Λ -morphism, φ , given by the composition

$$C^{n-1}(\tilde{K}) = {}^\omega \text{Hom}_\Lambda(C_{n-1}(\tilde{K}), \Lambda) \xrightarrow{\pi} F^{n-1}(\mathbf{C}(\tilde{K})) \xrightarrow{j} I \oplus P \xrightarrow{\begin{bmatrix} i & 0 \\ 0 & id \end{bmatrix}} \Lambda \oplus P$$

where π is the natural projection onto the cokernel, and $i: I \rightarrow \Lambda$ is the inclusion.

Since $F^{n-1}(\mathbf{C}(\tilde{K})) = C_{n-1}(\tilde{K})/\text{im}(d_{n-1}^*)$ by definition, $\varphi \circ d_{n-1}^* = 0$.

The dual of φ is a Λ -morphism

$$\varphi^*: (\Lambda \oplus P)^* \rightarrow C_{n-1}(\tilde{K})^{**}$$

As $C_{n-1}(\tilde{K}^{[n-1]})$ is a finitely generated free Λ -module, the natural map

$${}^\omega \varepsilon: C_{n-1}(\tilde{K}^{[n-1]}) \rightarrow C_{n-1}(\tilde{K}^{[n-1]})^{**}$$

is an isomorphism and we define

$$d_n: (\Lambda \oplus P)^* \rightarrow C_{n-1}(\tilde{K})$$

by

$$d_n = ({}^\omega \varepsilon)^{-1} \circ \varphi^*$$

By the naturality of ${}^\omega \varepsilon$,

$$\begin{aligned} d_{n-1} \circ d_n &= ({}^\omega \varepsilon)^{-1} \circ d_{n-1}^{**} \circ {}^\omega \varepsilon \circ ({}^\omega \varepsilon)^{-1} \circ \varphi^* \\ &= ({}^\omega \varepsilon)^{-1} \circ (\varphi \circ d_{n-1}^*)^* \\ &= 0 \end{aligned}$$

We first consider the case when P is free, so that $P \cong \Lambda^q$ for some $q \in \mathbb{N}$ and $\Lambda \oplus P \cong \Lambda^{q+1}$.

Since $\tilde{K}^{[n-1]}$ is $(n-2)$ -connected, the Hurewicz homomorphism

$$h_q: \pi_q(\tilde{K}^{[n-1]}) \rightarrow H_q(\tilde{K}^{[n-1]})$$

is an isomorphism for $q \leq n-1$ and we obtain the map

$$\begin{aligned} \varphi': \Lambda^{q+1} \cong (\Lambda \oplus P)^* &\rightarrow \ker(d_{n-1}) = H_{n-1}(\tilde{K}^{[n-1]}) \xrightarrow{h_{n-1}^{-1}} \pi_{n-1}(\tilde{K}^{[n-1]}) \\ x &\mapsto h_{n-1}^{-1}([d_n(x)]) \end{aligned}$$

Let \mathbf{C} be the chain complex of Λ -modules

$$\Lambda^{q+1} \cong (\Lambda \oplus P)^* \xrightarrow{d_n} C_{n-1}(\tilde{K}^{[n-1]}) \xrightarrow{d_{n-1}} \dots \rightarrow C_1(\tilde{K}^{[n-1]}) \rightarrow \Lambda$$

Then $Y = (\mathbf{C}, \overline{\varphi}, K^{[n-1]})$ is a homotopy system of order n .

As $C_i = 0$ for $i > n$, $\widehat{H}^{n+2}(Y; \Gamma_n Y) = 0$ and, by Proposition 8.3 in [2], there is a homotopy system $(\mathbf{C}, 0, X)$ of order $n+1$ realising Y , with X an n -dimensional CW -complex.

By construction, $\mathbf{C}(\tilde{X}) = \mathbf{C}$ and the universal cover of X is $(n-2)$ -connected.

The inclusion $i: K^{[n-1]} \rightarrow K$ extends to a map

$$f: X \rightarrow K = K(G, 1)$$

and we may consider $\omega \in H^1(K; \mathbb{Z}/2\mathbb{Z})$ as element of $H^1(X; \mathbb{Z}/2\mathbb{Z})$.

Proposition 4.1. *X is a PD^n -complex with fundamental triple (G, ω, μ) , that is*

- (i) $\mathbb{Z} \cong H_n(X; \mathbb{Z}^\omega) = \langle [X] \rangle$;
- (ii) $f_*([X]) = \mu$;
- (iii) $_ \cap [X]: H^r(X; {}^\omega \Lambda) \rightarrow H_{n-r}(X; \Lambda)$ is an isomorphism for every $r \in \mathbb{Z}$.

Proof. (i) As $\mathbf{C}(\tilde{X}) = \mathbf{C}$ is a chain complex of finitely generated free Λ -modules, the natural map

$$\eta_{\mathbf{C}}: \mathbb{Z}^{\omega} \otimes_{\Lambda} \mathbf{C} \longrightarrow \text{Hom}_{\Lambda}({}^{\omega}\text{Hom}_{\Lambda}(\mathbf{C}, \Lambda), \mathbb{Z})$$

is an isomorphism.

Hence, $H_n(X; \mathbb{Z}^{\omega}) = \ker(1 \otimes d_n) \cong \ker(\varphi^+)$, for

$$\begin{aligned} \varphi^+: \text{Hom}_{\Lambda}(\Lambda \oplus \Lambda^q, \mathbb{Z}) &\longrightarrow \text{Hom}_{\Lambda}({}^{\omega}\text{Hom}_{\Lambda}(C_{n-1}(\tilde{K}^{[n-1]}), \Lambda), \mathbb{Z}), \\ \psi &\longmapsto \psi \circ \varphi \end{aligned}$$

Since, by definition,

$$\varphi := \begin{bmatrix} i & 0 \\ 0 & id \end{bmatrix} \circ j \circ \pi: {}^{\omega}\text{Hom}_{\Lambda}(C_{n-1}(\tilde{K}), \Lambda) \longrightarrow \Lambda \oplus \Lambda^q$$

we obtain

$$\varphi^+ = \pi^+ \circ j^+ \circ \begin{bmatrix} i & 0 \\ 0 & id \end{bmatrix}^+$$

Since both π and j are surjective, both π^+ and j^+ are injective, whence

$$\begin{aligned} \ker(\varphi^+) &= \ker\left(\begin{bmatrix} i & 0 \\ 0 & id \end{bmatrix}^+\right) \\ &= \ker\left(\begin{bmatrix} i^+ & 0 \\ 0 & id \end{bmatrix}\right) \\ &\cong \ker(i^+) \end{aligned}$$

But I is generated by elements $1 - g$ ($g \in G$), and, for $\psi \in \text{Hom}_{\Lambda}(\Lambda, \mathbb{Z})$, we have

$$\begin{aligned} (\psi \circ i)(1 - g) &= \psi(1 - g) \\ &= \psi(1 - g.1) \\ &= \psi(1) - g.\psi(1) \\ &= 0 \end{aligned}$$

Hence,

$$\ker(\varphi^+) \cong \text{Hom}_{\Lambda}(\Lambda, \mathbb{Z}) \cong \mathbb{Z}$$

generated by

$$\text{aug} \circ pr_{\Lambda}: \Lambda \oplus \Lambda^q \longrightarrow \mathbb{Z}$$

the projection onto the first factor, followed by the augmentation.

Let $[X] = [1 \otimes x] \in H_n(X; \mathbb{Z}^{\omega})$ be the homology class corresponding to $\text{aug} \circ pr_{\Lambda}$ under the isomorphism $H_n(X; \mathbb{Z}^{\omega}) = \ker(1 \otimes d_n) \cong \ker(\varphi^+) \cong \text{Hom}_{\Lambda}(\Lambda, \mathbb{Z})$.

Then $x \in (\Lambda \oplus \Lambda^q)^*$ is projection onto the first factor.

(ii)

By the proof of Lemma 3.3, $\nu_{\mathbf{C}(\tilde{X}), n-1}([X])$ is represented by

$$F^{n-1}(\mathbf{C}(\tilde{X})) \longrightarrow I, \quad [\psi] \longmapsto \overline{\psi(d_n(x))}$$

Thus, given $\psi \in C_{n-1}(\tilde{X})^* = C_{n-1}(\tilde{K})^*$,

$$\begin{aligned}
\overline{\psi(d_n(x))} &= \overline{\psi(\omega \text{ev}^{-1}(\varphi^*(x)))} \\
&= \overline{\psi(\omega \text{ev}^{-1}(x \circ \varphi))} \\
&= \omega \text{ev}(\omega \text{ev}^{-1}(x \circ \varphi))(\psi) \\
&= (x \circ \varphi)(\psi) \\
&= (x \circ \begin{bmatrix} i & 0 \\ 0 & id \end{bmatrix} \circ j \circ \pi)(\psi) \\
&= (x \circ \begin{bmatrix} i & 0 \\ 0 & id \end{bmatrix} \circ j)([\psi]) \\
&= (i \circ pr_I \circ j)([\psi]) \\
&= h([\psi])
\end{aligned}$$

Hence, $\nu_{\mathbf{C}(\tilde{X}), n-1}([X])$ is the homotopy class of h so that

$$\begin{aligned}
\nu_{\mathbf{C}(\tilde{K}), n-1}(\mu) &= \nu_{\mathbf{C}(\tilde{X}), n-1}([X]) \\
&= \nu_{\mathbf{C}(\tilde{K}), n-1}(f_*([X]))
\end{aligned}$$

By Lemma 2.5 in [5], $\nu_{\mathbf{C}(\tilde{K}), n-1}$ is injective, whence $\mu = f_*([X])$.

(iii) Given $1 \leq i < n-1$

$$H_i(\tilde{X}; \Lambda) = H_i(\tilde{K}^{[n-1]}; \Lambda) = 0$$

and, by the definition of φ ,

$$H^{n-1}(\tilde{X}; \omega \Lambda^\omega) = 0$$

Moreover,

$$\begin{aligned}
H^i(\tilde{X}; \Lambda^\omega) &= H^i(\tilde{K}^{[n-1]}; \Lambda^\omega) \\
&\cong H^i(G; \Lambda^\omega) \\
&= 0
\end{aligned}$$

by hypothesis

Thus

$$_ \cap (1 \otimes [X]): H^i(X; \omega \Lambda) \longrightarrow H_{n-i}(X; \Lambda)$$

is an isomorphism for $0 < i < n$, and it remains only to consider $i = 0, n$.

As P and $\Lambda \oplus P$ are free, $\mathbf{C}(X)$ is a chain complex of free Λ -modules. Since the (twisted) evaluation map from a finitely generated free Λ -module to its double dual is an isomorphism, we obtain

$$\begin{aligned}
H^n(\tilde{X}; \omega \Lambda) &= H_{-n}(\omega \text{Hom}_\Lambda(\mathbf{C}(X), \omega \Lambda)) \\
&= \omega \text{Hom}_\Lambda(C_n(X), \omega \Lambda) / \text{im}(\varphi^*)^* \\
&= (\Lambda \oplus P)^{**} / \text{im}(\varphi^*)^* \\
&\cong \Lambda \oplus P / \text{im}(\varphi) \\
&\cong \Lambda / I \\
&\cong \mathbb{Z}
\end{aligned}$$

The class, $[\gamma]$, of the image of $(1, 0) \in \Lambda \oplus P$ under the (twisted) evaluation isomorphism generates $H^n(X; \omega \Lambda)$ and so, by Lemma 4.4 of [3],

$$\begin{aligned}
[\gamma] \cap [X] &= [\gamma] \cap [1 \otimes x] \\
&= \overline{[\gamma(x) \cdot e_0]} \\
&= [e_0]
\end{aligned}$$

as $\gamma(1 \otimes x) = 1$

where $e_0 \in C_0(X)$ is a chain representing the base point.

Thus,

$$_ \cap [X]: H^n(\tilde{X}; \omega \Lambda) \longrightarrow H_0(\tilde{X}; \Lambda)$$

is an isomorphism and $_ \frown (1 \otimes x)$ yields the chain homotopy equivalence

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \operatorname{im} d_{n-1}^* & \longrightarrow & C_{n-1}(X)^* & \longrightarrow & C_n(X)^* & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow _ \frown (1 \otimes x) & & \downarrow _ \frown (1 \otimes x) & & \downarrow \\ 0 & \longrightarrow & \operatorname{im} d_2 & \longrightarrow & C_1(X) & \longrightarrow & C_0(X) & \longrightarrow & 0 \end{array}$$

Applying the functor ${}^\omega \operatorname{Hom}_\Lambda(_, \Lambda)$, we obtain a chain homotopy induced by $(_ \frown (1 \otimes x))^*$. Hence, $(_ \frown (1 \otimes x))^*$ induces the isomorphism

$$H^0(X; \Lambda) \longrightarrow H_n(X; \Lambda)$$

By Lemma 2.1 in [3], $(_ \frown (1 \otimes x))^*$ induces an isomorphism in homology if and only if $_ \frown (1 \otimes x)$ does, whence

$$_ \frown [X]: H^0(X; {}^\omega \Lambda) \longrightarrow H_n(X; \Lambda)$$

is an isomorphism. □

Suppose now that P is projective, but not free.

Then there is a finitely generated Λ -module Q and a natural number q with $P^* \oplus Q \cong \Lambda^q$. The natural isomorphisms

$$\begin{aligned} (\Lambda \oplus P)^* \oplus \Lambda^\infty &\cong \Lambda^* \oplus P^* \oplus (Q \oplus P^* \oplus \dots) \\ &\cong \Lambda \oplus (P^* \oplus Q \oplus P^* \oplus Q \dots) \\ &\cong \Lambda^\infty \end{aligned}$$

show that $(\Lambda \oplus P)^* \oplus \Lambda^\infty$ is a free Λ -module.

Consider the chain complex \mathbf{D}

$$\begin{aligned} 0 \longrightarrow (\Lambda \oplus P)^* \oplus \Lambda^\infty &\xrightarrow{\begin{bmatrix} d_n & 0 \\ 0 & id \end{bmatrix}} C_{n-1}(\tilde{K}^{[n-1]}) \oplus \Lambda^\infty \xrightarrow{[d_{n-1} \ 0]} \\ &C_{n-2}(\tilde{K}^{[n-1]}) \xrightarrow{d_{n-2}} C_{n-3}(\tilde{K}^{[n-1]}) \longrightarrow \dots \end{aligned}$$

We may attach infinitely many n -balls to $K^{[n-1]}$ to obtain a CW -complex, K' , whose cellular chain complex coincides with the chain complex \mathbf{D} in dimensions $\leq n-1$.

Then $\tilde{K}'^{[n-1]}$ is $(n-2)$ -connected, and the Hurewicz homomorphism

$$h_q: \pi_q(\tilde{K}'^{[n-1]}) \longrightarrow H_q(\tilde{K}'^{[n-1]})$$

is an isomorphism for $q \leq n-1$. Defining the map

$$\begin{aligned} \varphi': (\Lambda \oplus P)^* \oplus \Lambda^\infty &\longrightarrow \ker(d_{n-1}) = H_{n-1}(\tilde{K}'^{[n-1]}) \xrightarrow{h_{n-1}^{-1}} \pi_{n-1}(\tilde{K}'^{[n-1]}) \\ x &\longmapsto h_{n-1}^{-1}([d_n(x)]) \end{aligned}$$

we obtain the homotopy system $Y' = (\mathbf{D}, \varphi, K'^{[n-1]})$ of order n .

As $D_i = 0$ for $i > n$, $\hat{H}^{n+2}(Y'; \Gamma_n Y') = 0$.

By Proposition 8.3 in [2], there is then a homotopy system $(\mathbf{C}, 0, X')$ of order $n+1$ realising Y' , with X' an n -dimensional CW -complex.

Observe that \mathbf{D} , the chain complex of X , is chain homotopy equivalent to the chain complex \mathbf{W}

$$\begin{aligned} \cdots \longrightarrow P^* \oplus Q &\xrightarrow{\begin{bmatrix} id & 0 \\ 0 & 0 \end{bmatrix}} P^* \oplus Q \xrightarrow{\begin{bmatrix} 0 & 0 \\ 0 & id \end{bmatrix}} P^* \oplus Q \xrightarrow{\begin{bmatrix} 0 & 0 \\ 0 & id \end{bmatrix}} \\ &(\Lambda \oplus P)^* \oplus Q \xrightarrow{\begin{bmatrix} d_n \\ 0 \end{bmatrix}} C_{n-1}(\widehat{K}^{[n-1]}) \xrightarrow{d_{n-1}} \\ &C_{n-2}(\widehat{K}^{[n-1]}) \xrightarrow{d_{n-2}} C_{n-3}(\widehat{K}^{[n-1]}) \longrightarrow \cdots \end{aligned}$$

By Theorem 2 of [7], there is a CW -complex, X , with cellular chain complex \mathbf{W} , homotopy equivalent to X' .

The proof that X realises (G, ω, μ) is similar to the proof of Proposition 4.1.

This completes the proof of Theorem A.

5. DECOMPOSITION AS CONNECTED SUM

Wall constructed a new PD^n complex from given ones using the *connected sum of PD^n complexes* (cf [8]). This allows PD^n complexes to be decomposed as connected sum of other, simpler PD^n complexes.

Take PD^n complexes $(X_k, \omega_k, [X_k])$ for $k = 1, 2$.

Then we may express X_k as the mapping cone

$$X_k = X'_k \cup_{f_k} e_k^n$$

for suitable $f_k: S^{n-1} \longrightarrow X'_k$.

Here, X'_k is an $(n-1)$ -dimensional CW -complex when $n > 3$, and when $n = 3$, X'_k is 3-dimensional with $H^3(X'_k; B) = 0$ for all coefficient modules, B .

For $k = 1, 2$, let $in_k: X'_k \longrightarrow X'_1 \vee X'_2$ be the canonical inclusion of the k^{th} summand and put

$$\widehat{f}_k := \iota_k \circ f_k: S^{n-1} \longrightarrow X'_1 \vee X'_2$$

so that \widehat{f}_k determines an element of $\pi_{n-1}(X'_1 \vee X'_2)$

Let $f_1 + f_2: S^{n-1} \longrightarrow X'_1 \vee X'_2$ represent the homotopy class $[\widehat{f}_1] + [\widehat{f}_2]$.

Then the connected sum, $X = X_1 \# X_2$, of X_1 and X_2 is the mapping cone of $f_1 + f_2$,

$$X_1 \# X_2 := (X'_1 \vee X'_2) \cup_{f_1 + f_2} e^n$$

It follows from the Seifert-van Kampen Theorem that

$$\pi_1(X) = \pi_1(X_1) * \pi_1(X_2)$$

The canonical inclusion $in_k: \pi_1(X_k) \longrightarrow \pi_1(X)$ induces a (left, resp. right) $\mathbb{Z}[\pi_1(X_k)]$ -module structure on any (left resp. right) $\Lambda = \mathbb{Z}[\pi_1(X)]$ -module. In particular Λ is a $\pi_1(X_k)$ -bimodule.

By the universal property of the free product, the morphisms $\omega_{X_k} = in_k^*(\omega_X)$ uniquely determine a morphism

$$\omega_X: \pi_1(X) \longrightarrow \mathbb{Z}/2\mathbb{Z}$$

Given left $\mathbb{Z}[\pi_1(X_k)]$ -modules M, N and a $\mathbb{Z}[\pi_1(X)]$ -morphism $\alpha: M \longrightarrow N$, put

$$\begin{aligned} L_k M &:= \Lambda \otimes_{\mathbb{Z}[\pi_1(X_k)]} M \\ L_k \alpha &:= id_{\Lambda} \otimes_{\mathbb{Z}[\pi_1(X_k)]} \alpha: \Lambda \otimes_{\mathbb{Z}[\pi_1(X_k)]} M \longrightarrow \Lambda \otimes_{\mathbb{Z}[\pi_1(X_k)]} N \end{aligned} \tag{*}$$

Plainly, L_k is a functor from the category of left $\mathbb{Z}[\pi_1(X_k)]$ -modules to the category of left Λ -modules.

Let $\mathbf{C}(\tilde{X})$ be the $\mathbb{Z}[\pi_1(X)]$ -cellular chain complex of the universal cover of X , and \mathbf{B} the subcomplex containing the n -cell attached by $f_1 + f_2$.

Then \mathbf{B} is a Poincaré duality chain complex ([2] p.2361) and it follows Theorem 2.3 of [3] that $L_k(\mathbf{C}(\tilde{X}_k))$ is also a Poincaré duality chain complex.

Let x be the chain representing the n -cell attached by $f_1 + f_2$. Repeated application of Theorem 2.3 of [3] shows that $L_1(\mathbf{C}(\tilde{X}_1)) + L_2(\mathbf{C}(\tilde{X}_2))$ is a Poincaré duality chain complex.

Hence $(X, \omega_X, [1 \otimes x])$ is a Poincaré duality complex. This is the *connected sum* of $(X_1, \omega_{X_1}, [X_1])$ and $(X_2, \omega_{X_2}, [X_2])$, introduced by Wall in [8].

We have seen that a necessary condition for a Poincaré duality complex to be a connected sum of Poincaré duality complexes is that its fundamental group be the free product of groups. We show that, in our context, this condition is also sufficient.

Theorem B. *A PD^n complex with $(n-2)$ -connected universal cover, decomposes as a connected sum if and only if its fundamental group decomposes as a free product of groups.*

Proof. It only remains to prove sufficiency.

Let $(X, \omega_X, [X])$ be a PD^n complex with $(n-2)$ -connected universal cover such that

$$\pi_1(X) = G_1 * G_2$$

with G_1, G_2 groups.

As $\pi_1(X)$ is finitely presentable, so are G_1 and G_2 .

For $j = 1, 2$, let $K_j = K(G_j; 1)$ be an Eilenberg-Mac Lane space with finite 2-skeleton.

Then $K_1 \vee K_2$ is an Eilenberg-Mac Lane space, $K(G_1 * G_2; 1)$, and

$$H_n(K; \mathbb{Z}^\omega) = H_n(K_1; \mathbb{Z}^{\omega_1}) \oplus H_n(K_2; \mathbb{Z}^{\omega_2})$$

where $\omega_j \in H^1(K_j; \mathbb{Z}/2\mathbb{Z})$ is the restriction of the orientation character $\omega \in H^1(K; \mathbb{Z}/2\mathbb{Z})$.

Thus, $\mu_X = \mu_1 + \mu_2$, with $\mu_j \in H_n(K_j; \mathbb{Z}^{\omega_j})$ for $j = 1, 2$.

By the discussion above, if the PD^n complex X_j , with $(n-2)$ -connected universal cover realises the fundamental triple (G_j, ω_j, μ_j) , then the connected sum of X_1 and X_2 realises the fundamental triple of X , whence, by the Classification Theorem in [2], X is orientedly homotopy equivalent to $X_1 \# X_2$.

Hence, it is sufficient to construct realisations of (G_j, ω_j, μ_j) ($j = 1, 2$).

Let L_j be the functor defined in (*), so that for $i \geq 1$

$$C_i(\tilde{K}) = L_1(C_i(\tilde{K}_1)) \oplus L_2(C_i(\tilde{K}_2))$$

It follows that

$$F^{n-1}(\mathbf{C}(\tilde{K})) = L_1(F^{n-1}(\mathbf{C}(\tilde{K}_1))) \oplus L_2(F^{n-1}(\mathbf{C}(\tilde{K}_2)))$$

and

$$I(\pi_1(X)) = L_1(I(G_1)) \oplus L_2(I(G_2))$$

where the canonical inclusion is given by

$$L_j(I(G_j)) \longrightarrow I(G), \quad \mu \otimes \lambda \longmapsto \mu\lambda$$

for $\mu \in \mathbb{Z}[\pi_1(X)]$ and $\lambda \in I(G)$ viewed as an element of $I(\pi_1(X))$.

Let $\varphi_j: F^{n-1}(\mathbf{C}(\tilde{K}_j)) \rightarrow I(G_j)$ be a $\mathbb{Z}[G_j]$ -morphism representing the class $\nu_{C(\tilde{K}_j), n-1}(\mu_j)$.

Then the class, $\nu_{C(\tilde{K}), i-1}(\mu)$ of homotopy equivalences is represented by

$$\begin{aligned} L_1(F^{n-1}(\mathbf{C}(\tilde{K}_1))) \oplus L_2(F^{n-1}(\mathbf{C}(\tilde{K}_2))) &= F^{n-1}(\mathbf{C}(\tilde{K})) \\ \downarrow L_1(\varphi_1) \oplus L_2(\varphi_2) & \\ L_1(I(G_1)) \oplus L_2(I(G_2)) &= I(G) \end{aligned}$$

and it follows from the proof of the analogous proposition for $n = 3$ on pp.269–270 in [5], that φ_j is a homotopy equivalence of modules.

By Theorem A, (G_j, ω_j, μ_j) is realised by a PD^n complex, X_j , with $(n-2)$ -connected universal cover. \square

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